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The Casimir effect upon a single plate

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Abstract

In the presence of an external field, the imposition of specific boundary conditions can lead to interesting new manifestations of the Casimir effect. In particular, it is shown here that even a single conducting plate may experience a non-zero force due to vacuum fluctuations. The origins of this force lie in the change induced by the external potential in the density of available quantum states.

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Externally imposed boundary conditions on a freely fluctuating electromagnetic field lead to the famous Casimir force between conducting surfaces separated by some small distance [1]. Recent interest has been stimulated by improvements in the ability to measure this force and many theoretical developments have resulted as well [2]. Of course, it is not necessary to consider just free fields. We could imagine a situation where the surfaces are embedded inside some classical external field (such as gravity). Although the interaction of electromagnetism with gravity is extremely weak, it may nevertheless be interesting to ask how the force between plates is changed by the external field. As it turns out, there are some non-trivial consequences. It will be shown in this paper that even a single surface can experience a net non-zero Casimir force under the influence of a linear external field.

As the simplest possible situation, consider a real scalar field $\phi(x)$ described by the Lagrangian $L = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}V(x)\phi^2$, where $V(x)$ is an externally prescribed field[1]. The Green's function $G(x, x')$ obeys $(\Box + V) = \delta^2(x - x')$. In this initial investigation $\mu = 0, 1$ only and $G(x, x', k)$, the Fourier-transformed Green's function, obeys

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2 - V(x)\right] G(x, x', k) = -\delta(x - x'). \tag{1}$$

The force is readily obtained as the space–space component of the canonical energy–momentum tensor $T^{\mu\nu}$,

$$T^{xx} = -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}k}{2\pi}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x'} + k^2 - V\right) G(x, x', k)|_{x=x'}. \tag{2}$$

[1] Jaffe and co-workers [4] have used $V(x)$ as a means to mock up the physical distribution of matter in conducting plates and address questions relating to conductivity at high frequencies. The purpose of introducing $V(x)$ in this paper is different. We note that Elizalde and Romeo [3] also considered a one-dimensional system perturbed by an external field. They did not, however, solve the system under the boundary conditions used in this paper.

If one sets $V = 0$ and imposes the Dirichlet condition at two points along the x-axis, $\phi(0) = \phi(a) = 0$, then the Green's function between the plates[2] for $a > x > x' > 0$ is immediately seen to be $\sin(kx')\sin[k(a-x)]\csc(ka)/k$. For the region $\infty > x' > x > a$, one wants outgoing waves as the boundary condition for equation (1) and so the appropriate Green's function is $\exp[\mathrm{i}k(x'-a)]\sin[k(x-a)]/k$. The positive exponential guarantees convergence once a rotation to the imaginary axis is made, $k \to \mathrm{i}K$, and equation (2) immediately yields the well-known result for the (attractive) force on the top plate[3],

$$T^{xx} = -\frac{\pi}{24a^2}. \tag{3}$$

Having reviewed the necessary formalism in a familiar context, let us now make a non-trivial choice for $V(x)$. By way of mocking up a constant force directed towards a fixed centre at $x = 0$, choose $V(x) = b|x|$ with $b > 0$ and $-\infty < x < \infty$. Intuitively speaking, as a scalar photon rises it loses energy and undergoes a redshift. The Dirichlet condition $\phi(a) = 0$ will be said to represent a single 'conducting plate' placed above the origin at a height a. For a translationally invariant potential, the forces on both sides of the plate would cancel. But, with a position-dependent potential, this would not be true. One can try to use perturbation theory in the 'coupling constant' b for computing the net force on the plate. Although this ultimately fails (for reasons to be discussed soon), it is nevertheless instructive to make an attempt.

At leading order in V, the solution to equation (1) is

$$G(x, x', k) = G_0(x, x', k) - \int \mathrm{d}y G_0(x, y, k)V(y)G_0(y, x', k), \tag{4}$$

where $G_0(x, x', k)$ is the Green's function for $V = 0$ and the appropriate range of arguments, together with boundary conditions corresponding to outgoing waves. A calculation for real k, followed by rotation to the imaginary K axis, yields the force just below and just above the plate at $x = a$,

$$T^{xx}_{\text{below}} = \int_0^\infty \frac{\mathrm{d}K}{2\pi}\left[-K + b\left(\frac{1 - 2Ka - 2\mathrm{e}^{-2Ka}}{4K^2}\right)\right], \tag{5}$$

$$T^{xx}_{\text{above}} = \int_0^\infty \frac{\mathrm{d}K}{2\pi}\left[-K - b\left(\frac{1 + 2Ka}{4K^2}\right)\right]. \tag{6}$$

The net force is

$$T^{xx} = T^{xx}_{\text{below}} - T^{xx}_{\text{above}} = b\int_0^\infty \frac{\mathrm{d}K}{2\pi}\,\frac{1 - \mathrm{e}^{-2Ka}}{2K^2}. \tag{7}$$

Although the linearly divergent integrals have cancelled, there is clearly an infrared divergence present as $K \to 0$. It is not hard to understand its origin: in arriving at equation (4) we have implicitly assumed that $k^2 > -|V(x)|$. Else, oscillatory solutions cannot exist. But, for a fixed k this condition is violated when x becomes sufficiently large and the unperturbed solution is wholly unsuitable. To make some sense of equation (7) one may think of cutting off the integral at the lower end with a value $K^2 \sim |a|b$ in which case $T^{xx} \sim |a|b$. Of course, one cannot take this result seriously since the use of perturbation theory is questionable, as is the imposition of an arbitrary infrared cut-off. Nevertheless, it is interesting to see that the force thus estimated is positive, increases with the distance of the plate away from the origin and is non-analytic in the strength of the external potential.

[2] The free Green's function for the other ordering is simply obtained from the symmetry $G(x, x') = G(x', x)$.

[3] For convenience, we shall frequently refer to the Dirichlet points as 'conducting plates' or 'plates'. The reader may wish to consult [2] for details leading to the result quoted here.

It is essential to solve the problem exactly. Fortunately, for the simple potential we have chosen this is possible. Only the Green's function near the plate at $x = a$ (with $a > 0$) is needed. To proceed, first consider the region for $0 < a < x' < x$. Define a Euclidean dimensionless momentum κ, $k = \mathrm{i}b^{1/3}\kappa$. Equation (1) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y\right] G(y, y', \kappa) = -b^{-1/3}\delta(y - y'), \tag{8}$$

$y = \kappa^2 + (x/a)\eta^{\frac{1}{3}}$ where $\eta = ba^3$. Both y and η are positive and dimensionless. The solutions of $G'' + yG = 0$ are the Airy functions, $Ai(y)$ and $Bi(y)$, and the outgoing wave condition requires that $Bi(y)$ be excluded for $x > a$. The Green's function in this region is

$$\pi a \eta^{-1/3} Ai\left(\kappa^2 + \frac{x}{a}\eta^{\frac{1}{3}}\right) \frac{Ai\left(\kappa^2 + \eta^{\frac{1}{3}}\right) Bi\left(\kappa^2 + \frac{x'}{a}\eta^{\frac{1}{3}}\right) - Ai\left(\kappa^2 + \frac{x'}{a}\eta^{\frac{1}{3}}\right) Bi\left(\kappa^2 + \eta^{\frac{1}{3}}\right)}{Ai\left(\kappa^2 + \eta^{\frac{1}{3}}\right)}. \tag{9}$$

From this and equation (2), T^{xx}_{above} follows,

$$T^{xx}_{\text{above}} = \frac{\eta^{2/3}}{a^2} \int_0^\infty \frac{\mathrm{d}\kappa}{2\pi} \frac{Ai'\left(\kappa^2 + \eta^{\frac{1}{3}}\right)}{Ai\left(\kappa^2 + \eta^{\frac{1}{3}}\right)} \tag{10}$$

$$= \frac{\eta^{2/3}}{a^2} \int_0^\infty \frac{\mathrm{d}\kappa}{2\pi} \left[-\kappa - \frac{\eta^{\frac{1}{3}}}{2\kappa} - \frac{1}{4\kappa^2} - \cdots\right]. \tag{11}$$

In calculating the force on the other side of the origin, one needs to recognize that the arguments of the Airy functions change into $y = \kappa^2 - (x/a)\eta^{\frac{1}{3}}$ for negative x. Again, the outgoing wave condition requires that $Bi(y)$ be excluded for $x < 0$. Finally, one requires continuity of the solution and derivative at $x = 0$, as well as the jump condition imposed by the delta function. This yields the Green's function, from which the force below the plate is calculated to be

$$T^{xx}_{\text{below}} = \frac{\eta^{2/3}}{a^2} \int_0^\infty \frac{\mathrm{d}\kappa}{2\pi} \times \frac{2Ai(\kappa^2)Ai'(\kappa^2)Bi'\left(\kappa^2 + \eta^{\frac{1}{3}}\right) - Ai'\left(\kappa^2 + \eta^{\frac{1}{3}}\right)\left(Ai'(\kappa^2)Bi(\kappa^2) + Ai(\kappa^2)Bi'(\kappa^2)\right)}{Ai\left(\kappa^2 + \eta^{\frac{1}{3}}\right)\left(Ai'(\kappa^2)Bi(\kappa^2) + Ai(\kappa^2)Bi'(\kappa^2)\right) - 2Ai(\kappa^2)Ai'(\kappa^2)Bi\left(\kappa^2 + \eta^{\frac{1}{3}}\right)} \tag{12}$$

$$= \frac{\eta^{2/3}}{a^2} \int_0^\infty \frac{\mathrm{d}\kappa}{2\pi} \left[-\kappa - \frac{\eta^{\frac{1}{3}}}{2\kappa} + \frac{1}{4\kappa^2} - \cdots\right]. \tag{13}$$

The expansion of the integrand above is for large κ. Although T^{xx}_{above} and T^{xx}_{below} are separately divergent at the upper limit (as might be expected from the infinite pressure of photons striking each surface), $T^{xx} = T^{xx}_{\text{below}} - T^{xx}_{\text{above}}$ is finite. The integrals must be done numerically. Reinstating $\hbar$ and c can be expressed as

$$T^{xx} = \hbar c \frac{\eta^{2/3}}{a^2} f(\eta). \tag{14}$$

In figure 1, we plot T^{xx} as a function of η for $a = 1$. The force vanishes at $a = 0$, monotonically increases with a and is repulsive. In fact, expanding the difference of equation (10) and equation (12) leads to $f(\eta) \sim \eta^{\frac{1}{3}}$ which shows that the force vanishes linearly with a.

In summary, it has been shown here that one can expect even a single conducting plate placed in the vacuum to experience a net quantum force. The force has the same origin

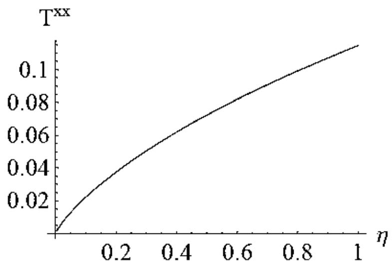

Figure 1. The Casimir force for fixed distance ($a = 1$) as a function of the coupling strength b.

as the Casimir effect, i.e. is a manifestation of the zero-point fluctuations of a quantum field. The difference in the density of normal modes above and below the plate, induced by the position-dependent external potential, is the responsible mechanism. The present investigation was performed with a simple, real, scalar field but one expects a similar effect for the electromagnetic field (or any other field) as well.

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Note added. Subsequent to the completion of the research reported here, a more extensive treatment of the effect of vacuum fluctuations upon a single boundary was undertaken by R L Jaffe and A Scardicchio on 'Casimir buoyancy' [5].

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